International Journal of Theoretical Physics, Vol. 28, No. 2, 1989

Entropy Balance in "Pure" Interactions of Open Quantum Systems

R. Urigu[1]

Received August 10, 1988

Processes are considered in which a statistical ensemble ω of quantum systems is split into ensembles, or channels (ω_i), conditional to the occurrence, with respective probabilities (p_i^ω), of associated macroscopic effects. These processes are described here by a family of operations $T_i: \omega \to p_i^\omega \omega_i^{\mathcal{T}}$, which remarkably generalize the usual "state reductions" of the nondestructive measurements. In a previous work it was proved that the microscopic entropy of the given open system decreases or at most remains constant if all the T_i are pure operations, i.e., they transform pure states into pure states; it is proved here that the increase in entropy of the external world, computed as $S^{\mathcal{T}m}(\omega) = -\sum_i p_i^\omega \lg p_i^\omega$, is sufficient to compensate for such an entropy decrease whenever the T_i are all pure operations of the first kind, whereas whenever some T_i is pure of the second kind (or nonpure, too), the total entropy, computed as above, may decrease.

1. INTRODUCTION

In the quantum theory of open systems conditional state changes, called operations, are considered which remarkably generalize the Hamiltonian evolution of the closed systems (Haag and Kastler, 1964; Davies, 1976).

In the following we are concerned with a kind of physical process in which an original ensemble ω of quantum systems is split, by an external intervention, into a countable family of ensembles (or channels) $(\omega_i^{\mathcal{T}})$:

$$\omega \to (\omega_1^{\mathcal{T}}, \omega_2^{\mathcal{T}}, \ldots) \tag{1}$$

Precisely, such a splitting requires the production, by the quantum open system, of some macroscopic effects (Ludwig, 1983) on the external world; afterward the transmission of ω into the final channels occurs,

[1]Dipartimento di Matematica, Politecnico di Torino, 10100 Torino, Italy.

0020-7748/89/0200-0147$06.00/0

according to such effects, with respective probabilities (p_i^ω), so that $\forall\omega$, $\sum_i p_i^\omega = 1$.

Here a somewhat phenomenological description of such processes is considered which consists of a countable family $\mathcal{T} = (T_i)$ of operations:

$$\mathcal{T} = (T_i), \qquad T_i: \omega \to p_i^\omega \omega_i^{\mathcal{T}}; \qquad i = 1, 2, \ldots \tag{2}$$

An interaction process, as outlined above and described in this way, will be called a *discrete external interaction* of a quantum open system.

Well-known examples of discrete external interactions refer to nondestructive measurements, in which the operations of the various channels are expressed by an appropriate reduction postulate [e.g., Lüder's postulate for the "perfect" nondestructive measurement (Lüders, 1951; Ludwig, 1983, Chapter 17)]; the irreversible character of such processes was early recognized and discussed by von Neumann (1955, Chapter 5), who devised to this purpose, through thermodynamic arguments, the concept of entropy for quantum systems.

Following this approach, as the initial state ω is transmitted into a final channel, conditional to the occurrence of the respective exit of the measurement (effect), a well-suited entropy concept for the measured system appears to be the conditional entropy (Shannon and Weaver, 1949) $S^{\mathcal{T}c}(\omega) = \sum_i p_i^\omega S(\omega_i^{\mathcal{T}})$, which is the mean value of the von Neumann entropies $S(\omega_i^{\mathcal{T}})$ of the different channels.

Concerning the measuring apparatus, it seems quite natural to identify its final entropy with the "mixing entropy" (Wehrl, 1978) $S^{\mathcal{T}m}(\omega) = -\sum_i p_i^\omega \lg p_i^\omega$, i.e., the Shannon entropy of the final distribution probability (p_i^ω) of the "pointer" (Shannon and Weaver, 1949), the initial entropy being assumed to be zero.

It is worthwhile to remark that, within the foregoing assignment of final entropies in a measurement process, the final state of the apparatus turns out to be equivalently represented by the classical discrete probability distribution (p_i^ω). This description of the apparatus may be justified to some extent in the quantum theory of measurement: in particular, in a recent paper of Herbut (1986), concerning an improvement of Jauch's approach, the compound state of the quantum system plus the measuring apparatus is described by a "hybrid" state $(\omega_i^{\mathcal{T}}, p_i^\omega)$.

Now, due to well-known inequalities (Klein, 1931; von Neumann, 1955; Shannon and Weaver, 1949; Wehrl, 1978), it turns out that in any perfect nondestructive measurement

$$\forall\omega, \qquad S(\omega) \leq S^{\mathcal{T}c}(\omega) + S^{\mathcal{T}m}(\omega) \tag{3}$$

which means that, according to the above assignment of entropies, the total entropy cannot decrease. [Mathematically, (3) expresses the result stated by von Neumann; however he, as well as most authors, makes a different distribution of entropies, namely the whole entropy $S(\sum_i p_i^\omega \omega_i^{\mathcal{T}}) = S^{\mathcal{T}c}(\omega) + S^{\mathcal{T}m}(\omega)$ is assigned to the measured system (von Neumann, 1955, Chapter 6)].

In a previous work (Ascoli and Urigu, 1984), another inequality has been proved, namely

$$\forall \omega, \qquad S^{\mathcal{T}c}(\omega) \leq S(\omega) \tag{4}$$

which states that the entropy of the quantum system alone does decrease or at most remains constant. Now it turns out that the proof holds not only for the perfect nondestructive measurement, but for any discrete external interaction $\mathcal{T} = (T_i)$ which is "pure," i.e., such that, for any i, T_i is a pure operation, i.e., it transforms pure states into pure states.

So, one naturally wonders whether (3) is true for this more general class of processes, too.

It turns out that the known results do not apply so directly as for the measuring process considered above: in this respect there proves useful the known classification of the abstract pure operations into two classes, first kind and second kind operations (Davies, 1969, 1976; Hellwig and Kraus, 1969; Haag and Kastler, 1964). In fact, examples are given in Section 4 which show that (3) is contradicted whenever some pure operation of the second kind occurs within $\mathcal{T}$ (or, more generally, when nonpure operations occur).

On the other hand, it is proved in Section 3 that (3) holds true for any pure discrete interaction $\mathcal{T} = (T_i)$ such that, for any i, T_i is a pure operation of the first kind. Thus, in this case, if the increase of external entropy is identified with the entropy $S^{\mathcal{T}m}(\omega)$, an entropy decrease for the quantum system always occurs together with an entropy production such that the entropy of the whole system including the surroundings cannot decrease, preventing therefore the possible existence of such engines as the so-called "Maxwell's demon" (Szilard, 1929; Brillouin, 1962; Penrose, 1970).

It may finally be of interest to remark that the mathematical construction used in the proof provides an alternative, less phenomenological model of the discrete external interactions of open systems. Such a model arises from a quantum description of the surroundings along the lines of the work of Hellwig and Kraus (1969, 1970) and it is essentially equivalent to Kraus's description (1983, Section 5) of two "complementary operations." The discussion of Section 4 concerning such a quantum model leads to some understanding of the fact that only in the case of an external interaction of the first kind may the external world equivalently be described by the classical state (p_i^ω), so that its final entropy may be identified with $S^{\mathcal{T}m}(\omega)$.

2. PRELIMINARY CONCEPTS

In the following the state ω of the quantum system is described by a density operator W of some separable Hilbert space H; let $\mathcal{W}$ denote the set of all density operators in H and $\mathcal{W}^L$ the real vector space generated by $\mathcal{W}$.

A physical operation is described here as usual (Haag and Kastler, 1964; Ludwig, 1970, 1983; Davies, 1976) by a linear mapping T from $\mathcal{W}^L$ into itself which is positive and does not increase the trace-norm: i.e., for any $W \in \mathcal{W}^L$ such that $W \geq 0$ one has $T(W) \geq 0$ and $\text{Tr } W \geq \text{Tr}[T(W)] \geq 0$. The transmission probability p^ω of the initial state ω through T is given by $p^\omega = \text{Tr}[T(W)]$ and, in the case $p^\omega \neq 0$, the normalized operator $T(W)/p^\omega$ describes the final transmitted state.

The main interest in the sequel is in those special operations, called pure operations, which transform pure states into pure states (Davies, 1976; Hellwig and Kraus, 1969); it has to be pointed out that, according to this terminology, a pure operation is not necessarily an extremal element of the convex set of the operations [see Remark 1 to (7) below] (Davies, 1976, p. 21).

A theorem due to Davies (1969; 1976, Theorem 3.1, p. 21) allows a useful classification of the pure operations, defined as above. The statement may be formulated, with reference to the terminology of Hellwig and Kraus (1969), as follows.

Theorem. Every map T describing a pure operation within the state space $\mathcal{W}^L$ is of one, and only one, of the two following kinds:

1. *Pure operations of the first kind:*

$$T(W) = AWA^* \tag{5}$$

where A is a bounded linear or antilinear operator of H with $\|A\| \leq 1$ [the adjoint A^* of any antilinear bounded operator A of H is defined by (see, e.g., Messiah, 1961-1962) $\forall|\xi\rangle, |\psi\rangle \in H, \langle A^*\xi, \psi\rangle = \langle \xi, A\psi\rangle$].

2. *Pure operations of the second kind:*

$$T(W) = \text{Tr}(WB)|\psi\rangle\langle\psi| \tag{6}$$

where $|\psi\rangle$ is a normalized vector of H, and B is a linear operator of H such that $0 \leq B \leq 1_H$ and $\dim(BH) \geq 2$, i.e., its range is at least a two-dimensional subspace of H.

An operation having the form (6), even in the case B has a one-dimensional range, is called a degenerate operation (Davies, 1976); we remark that a degenerate operation may always be represented in the form

$$\forall W \in \mathcal{W}^L, \qquad T(W) = \text{Tr}(WB)|\psi\rangle\langle\psi| = \sum_i A_i W A_i^* \tag{7}$$

where each A_i may be chosen to be either a linear or an antilinear bounded operator of H. Take, e.g., for any i, $A_i = |\psi\rangle\langle\phi_i\sqrt{B}|$, where (ϕ_i) is an orthonormal basis in H and $\sqrt{B}$ is the positive square root of B (the sum converges with respect to the trace-norm topology); every linear A_i can instead be separately replaced by the antilinear KA_i, where K is any antilinear conjugation operator which leaves $|\psi\rangle$ invariant (see Section 3).

Remark 1. Starting from the above construction, it may be proved that the pure operation (6) is a convex linear combination of pure operations of the first kind if and only if $\mathrm{Tr}\, B \leq 1$.

Remark 2. It is easy to see that one and the same operation of the first kind (5) can be expressed either through a linear or an antilinear operator A if and only if it is a degenerate operation.

The pure operations under Theorem 1 do not have common elements, according to the following proposition:

Proposition. Let A, B be bounded linear operators of H with $B \geq 0$; $|\psi\rangle \in H$. Then the following conditions are equivalent:

(a) $\forall W \in \mathcal{W}^L, \mathrm{Tr}(WB)|\psi\rangle\langle\psi| = AWA^*$.

(b) B has a one-dimensional range.

Proof. (b)$\Rightarrow$(a): Take, in the above construction (7), ϕ_1 as a normalized eigenvector of B belonging to its one-dimensional range; then the sum on the right-hand side of (7) reduces to one term only.

(a)$\Rightarrow$(b): for any $|\xi\rangle \in H$, take $W = |\xi\rangle\langle\xi|$; then

$$\forall |\xi\rangle \in H, \qquad \langle\xi, B\xi\rangle|\psi\rangle\langle\psi| = |A\xi\rangle\langle\xi A^*|$$

Thus for any $|\xi\rangle \in H$, $A|\xi\rangle = \alpha(\xi)|\psi\rangle$ with $\alpha(\xi) \in C$; then $\dim(AH) = 1$, hence $\dim(A^*H) = 1$; therefore, as $B = A^*A$, $\dim(BH) = 1$ ■

The physical processes called in the introduction *discrete external interactions* will be described, as in (2), by a finite or infinite countable family $\mathcal{T} = (T_i)$ of operations on $\mathcal{W}^L$ such that

$$\forall \omega \in \mathcal{W}, \qquad \sum_i p_i^\omega = \sum_i \mathrm{Tr}[T_i(W)] = 1 \tag{8}$$

This concept corresponds to Davies' definition of "instrument" on a discrete space (Davies, 1976). For any i such that $p_i^\omega \neq 0$, $\omega_i^{\mathcal{T}} = (1/p_i^\omega)T_i(\mathrm{W})$ describes the ith channel.

A discrete external interaction $\mathcal{T} = (T_i)$ such that, for any i, T_i is a pure operation of the first kind will be called a *pure discrete (external) interaction of the first kind.*

Let us finally introduce the required entropy concepts.

The quantum-theoretical (microscopic) entropy of an ensemble ω described by a density operator W is expressed by the von Neumann formula (von Neumann, 1955)

$$S(\omega) = \operatorname{Tr} \eta(W) \tag{9}$$

where $\eta = \eta(x)$ is the real, continuous concave function $\eta(x) = -x \lg x$ for $x > 0$, $\eta(0) = 0$.

With reference to a discrete external interaction $\mathcal{T}$, as described above, we shall be concerned also with the following entropy concepts:

$S_i^{\mathcal{T}}(\omega) = S(\omega_i^{\mathcal{T}})$, for any i such that $p_i^{\omega} \neq 0$: *microscopic entropy of the ith channel.*

$$S^{\mathcal{T}c}(\omega) = \sum_{\substack{i \\ p_i^{\omega} \neq 0}} p_i^{\omega} S_i^{\mathcal{T}}(\omega)$$

is the *final microscopic entropy of* the quantum system, that is, the mean entropy of the channels or *conditional entropy* (Shannon and Weaver, 1949).

$S^{\mathcal{T}m}(\omega) = \sum_i \eta(p_i^{\omega})$: *mixing entropy* or *Shannon's information entropy* of the probability distribution (p_i^{ω}) (Wehrl, 1978, Section II.B; Shannon and Weaver, 1949).

It is worthwhile to remark that the sum of entropies $S^{\mathcal{T}c} + S^{\mathcal{T}m}$ may be expressed as the von Neumann entropy of a suitable density operator in a larger Hilbert space $\hat{H}$. In fact, take for $\hat{H}$ the direct sum of a number of copies of H equal to the cardinality of the family $\mathcal{T} = (T_i)$ of operations: $\hat{H} = H \oplus H \oplus H \oplus \cdots$ Then it is easily checked, for instance, by diagonalizing the operator $\oplus_i T_i(W)$, that (Wehrl, 1978, Section II.F)

$$\forall \omega \in \mathcal{W}, \qquad S^{\mathcal{T}c}(\omega) + S^{\mathcal{T}m}(\omega) = \sum_i p_i^{\omega} \operatorname{Tr}\{\eta[T_i(W)/p_i^{\omega}]\} + \sum_i \eta(p_i^{\omega})$$

$$= S\left[\bigoplus T_i(W)\right] \tag{10}$$

3. ENTROPY BALANCE IN A PURE DISCRETE INTERACTION OF THE FIRST KIND

Let us firstly restate the theorem proved in a previous paper (Ascoli and Urigu, 1984).

Theorem 1. Let $\mathcal{T} = (T_i)$ describe a discrete external interaction (see Section 2); then the following two conditions are equivalent:

(i) $\mathcal{T}$ is pure.

(ii) $\forall \omega \in \mathcal{W}$, $S^{\mathcal{T}c}(\omega) \leq S(\omega)$.

Proof. The implication (i)$\Rightarrow$(ii) is proved in Ascoli and Urigu (1984).

To prove (ii)$\Rightarrow$(i), let us consider any nonpure interaction $\mathcal{T} = (T_i)$; then there exist a channel i and a pure state ω such that T_i transforms ω into a mixed state $\omega_i^{\mathcal{T}}$. Therefore $S_i^{\mathcal{T}}(\omega) \equiv S(\omega_i^{\mathcal{T}}) > S(\omega) = 0$ and $S^{\mathcal{T}c}(\omega) > S(\omega)$, contrary to the hypothesis (ii). ■

Let us now prove the main theorem of this paper.

Theorem 2. Let $\mathcal{T} = (T_i)$ describe a pure discrete interaction of the first kind; then the following inequality holds:

$$\forall \omega \in \mathcal{W}, \qquad S(\omega) \leq S^{\mathcal{T}c}(\omega) + S^{\mathcal{T}m}(\omega) \tag{11}$$

Proof. As $\mathcal{T} = (T_i)$ consists of operations of the first kind, the T_i are described by

$$\forall i, \forall W \in \mathcal{W}^L, \qquad T_i(W) = A_i W A_i^* \tag{12}$$

where (A_i) is a family of bounded linear or antilinear contractions in H fulfilling the normalization condition (8), i.e. (with respect to the ultraweak topology of operators),

$$\sum_i A_i^* A_i = 1_H \tag{13}$$

Consider first the case in which every A_i is linear.

According to well-known results (Riesz and Nagy, 1960), any linear contraction A in a Hilbert space H can be expressed, within an "extension" space $\hat{H} \supset H$, as the first diagonal element

$$A = \mathcal{U}_{11}$$

of the operator matrix representing a unitary operator $\mathcal{U}$ of $\hat{H}$, with respect to the decomposition $\hat{H} = H \oplus (\hat{H} \ominus H)$.

Let us show that here, thanks to condition (13), the whole family (A_i) can be expressed within the extension space $\hat{H}$, defined at the end of the previous section, as the whole first column

$$(A_i) = (\mathcal{U}_{i1}) \tag{14}$$

of the operator matrix representing an operator $\mathcal{U}$ of $\hat{H}$, which is isometric on $H \oplus 0 \oplus 0 \oplus \cdots$. A representation of the type (14) can be explicitly derived by a straightforward generalization of well-known procedures (Riesz and Nagy, 1960; Hellwig and Kraus, 1969, 1970; Kraus, 1971); an essentially equivalent representation is found in Kraus (1983, Section 5).

Let H' be a Hilbert space with dimension equal to the cardinality of the family (T_i) of the operations to be represented. Let (Q_i) be a maximal orthogonal family of one-dimensional projectors of H'; then the tensor product space $\hat{H} = H \otimes H'$ may be decomposed as

$$\hat{H} = H \otimes H' = \sum_i (H \otimes Q_i H') = \sum_i H_i \tag{15}$$

where all the H_i are canonically identifiable with H. Let now $\mathcal{A}$ be the column matrix constructed with the operators $A_2, A_3, \ldots$, and let us define the following matrix of operators of H, which represents, with reference to the decomposition (15), an operator $\mathcal{U}$ of $\hat{H}$:

$$\mathcal{U} \equiv \begin{pmatrix} A_1 & U_1 A^* \\ \mathcal{A} & -\sqrt{(1_{\hat{H} \ominus H} - \mathcal{A}\mathcal{A}^*)} \end{pmatrix} \tag{16}$$

where U_1 is the partially isometric operator, which enters into the polar representation $A_1 = U_1 \sqrt{(A_1^* A_1)}$.

The operator $\mathcal{U}$ defined in this way is then a partially isometric operator whose initial projector $E = \mathcal{U}^* \mathcal{U}$ is such that $E\hat{H} \supseteq H \oplus 0 \oplus 0 \ldots$ (Riesz and Nagy, 1960) and it describes the whole family of linear contractions (A_i) within a Hilbert space $\hat{H}$ whose canonical decomposition is minimal, in the sense that the component subspaces are in one-to-one correspondence to the operations of $\mathcal{T}$. Whenever $\mathcal{U}_1$ may be chosen to be unitary (as always it may be in the finite-dimensional case, U, too, is unitary. A unitary U may always be obtained, in analogy to Kraus construction (Kraus, 1983, Section 5) of two "complementary" operations, by adding within $\hat{H}$ an extra component subspace H; then this U is simply obtained from (16) by taking $A_1 = 0$ and $U_1 = 1_H$.

Then, as is easily checked, the family of operations (12) can be represented as

$$\forall i, \forall W \in \mathcal{W}^L, \qquad T_i(W) = A_i W A_i^* = \mathrm{Tr}'[(1 \otimes Q_i)\mathcal{U}(W \otimes Q_1)\mathcal{U}^*(1 \otimes Q_i)] \tag{17}$$

where Tr′ means the partial trace with respect to H'; furthermore,

$$\forall W \in \mathcal{W}^L, \qquad \bigoplus_i T_i(W) = \sum_i (1 \otimes Q_i)\mathcal{U}(W \otimes Q_1)\mathcal{U}^*(1 \otimes Q_i) \tag{18}$$

We may now apply a well-known inequality concerning the entropy variation in a perfect nondestructive measurement (Klein, 1931; von Neumann, 1955; Wehrl, 1978) to obtain

$$\forall W \in \mathcal{W}, \qquad S\left[\sum_i (1 \otimes Q_i)\mathcal{U}(W \otimes Q_1)\mathcal{U}^*(1 \otimes Q_i)\right] \geq S[\mathcal{U}(W \otimes Q_1)\mathcal{U}^*] \tag{19}$$

Furthermore, by the unitary invariance property of the von Neumann entropy (9) and by its additivity property (Wehrl, 1978, Section II.E.), one obtains

$$\forall W \in \mathcal{W}, \qquad S[\mathcal{U}(W \otimes Q_1)\mathcal{U}^*] = S(W \otimes Q_1) = S(W) \tag{20}$$

Finally, through the identity (10) of the previous section and through (18)-(20), the inequality (11) is proved for first-kind interactions (12) with linear A_i.

The case in which some of the A_i are antilinear can be reduced to the one just considered as follows.

For any orthonormal basis (e_i) of H, let K be the antilinear operator that transforms the components of any vector, with respect to (e_i), into their complex conjugates; then any antilinear operator A can be represented in the form $A = K(KA)$, KA being a linear operator. We also use the known properties concerning adjoints of antilinear operators (Messiah, 1961-1962): $K^* = K$ and $(AB)^* = B^*A^*$, A and B being either linear or antilinear operators. Then, for any self-adjoint linear operator B of trace class of H,

$$\begin{aligned}\mathrm{Tr}(KBK) &= \sum_i \langle e_i, KBKe_i\rangle = \sum_i \overline{\langle e_i K^*, BKe_i\rangle} \\ &= \sum_i \overline{\langle e_i, Be_i\rangle} = \sum_i \langle e_i, Be_i\rangle = \mathrm{Tr}\, B \end{aligned}\tag{21}$$

In general, for any function f such that $f(B)$ remains of trace class, using the spectral resolutions $B = \sum_i b_i P_i$, $KBK = \sum_i b_i KP_iK$, hence $f(KBK) = \sum_i f(b_i) KP_iK$, it follows that

$$\mathrm{Tr}[f(KBK)] = \sum_i f(b_i)\,\mathrm{Tr}(KP_iK) = \sum_i f(b_i)\,\mathrm{Tr}\,P_i = \mathrm{Tr}[f(B)]$$

Therefore, for a channel with antilinear A_i,

$$p_i^{\omega} = \mathrm{Tr}(A_i W A_i^*) = \mathrm{Tr}[K(KA_i)W(KA_i)^*K^*] = \mathrm{Tr}[(KA_i)W(KA_i)^*]$$

and likewise

$$S_i^{\mathcal{T}}(\omega) = \mathrm{Tr}\, s[K(KA_i)W(KA_i)^*K^*/p_i^{\omega}] = \mathrm{Tr}\, s[(KA_i)W(KA_i)^*/p_i^{\omega}]$$

Thus, the sum $S^{\mathcal{T}c} + S^{\mathcal{T}m}$ remains unchanged when the antilinear operators occurring within (A_i) are replaced by their respective linear parts (KA_i), so that the above proof applies.

Remark. Though Theorem 1 remains true when substituting in the definitions of the entropies the function $s(x)$ with any continuous, concave function $f(x)$ such that $f(0) = f(1) = 0$, this is not the case for Theorem 2. In fact, the von Neumann entropy (9) is the only functional of ω, up to a constant factor, which satisfies the identity (10), i.e., which has the additivity property (see Wehrl, 1978, Section II.E; Shannon and Weaver, 1949).

4. CONCLUDING REMARKS

According to the main theorem proved above and to the one proved in (Ascoli and Urigu (1984), for any pure discrete external interaction of the first kind

$$\forall \omega \in \mathscr{W}, \qquad S^{\mathscr{T}c}(\omega) \leq S(\omega) \leq S^{\mathscr{T}c}(\omega) + S^{\mathscr{T}m}(\omega) \tag{22}$$

As remarked in advance in the Introduction, the right-hand inequality of (22) cannot hold unconditionally in the case $\mathscr{T}$ describes a pure interaction which involves operations of the second kind. Let us take, as a typical example, the following interaction $\mathscr{T}$, which gives rise to a single channel through a single operation T of the second kind:

$$\mathscr{T} = (T), \qquad T\colon\ W \to T(W) = (\text{tr } W)P \tag{23}$$

where P is a one-dimensional projector of H. Then, for any nonpure state $\omega_m \in \mathscr{W}$: $S(\omega_m) > S^{\mathscr{T}c}(\omega_m) + S^{\mathscr{T}m}(\omega_m) = 0$.

For interactions which involve nonpure operations, too, in spite of their mixing-enhancing property, the right-hand inequality of (22) is in general not true. Consider, e.g., with $H = C^2$, the following interaction, with a single channel, depending on one real parameter α:

$$\mathscr{T}(\alpha) = (T_\alpha), \qquad T_\alpha\colon \quad W \to T_\alpha(W) = PWP + U(\alpha)P^\perp WP^\perp U^{-1}(\alpha) \tag{24}$$

where $P + P^\perp = 1_{C^2}$ and $U(\alpha)$ is a unitary operator whose representation, with respect to an orthonormal basis associated with $(P, P^\perp)$, is

$$U(\alpha) = \begin{pmatrix} \cos(\alpha/2) & \text{sen}(\alpha/2) \\ -\text{sen}(\alpha/2) & \cos(\alpha/2) \end{pmatrix}, \qquad \alpha \text{ real}$$

Formula (24) may describe a Stern-Gerlach experiment, with spin-$\frac{1}{2}$ particles, in which the two emerging beams are mixed, after a rotation over an angle α of the polarization of the lower beam.

For $\alpha = \pi$, formula (24) gives an equivalent expression for (23) [see also (7)]; taking instead $\alpha \neq \pi$, but close to π, the initial state W is in general transformed into a mixed state which is in any case "close" to the pure state P, so that $S^{\mathscr{T}c}(\omega) + S^{\mathscr{T}m}(\omega) \approx 0$; therefore, for the chaotic initial state $W_c = \frac{1}{2}P + \frac{1}{2}P^\perp$, one has $S(W_c) = \lg 2 > S^{\mathscr{T}c}(W_c) + S^{\mathscr{T}m}(W_c)$.

This example also shows that in general a pure interaction of the second kind may be depicted as a limiting case of a nonpure interaction.

According to the above considerations, if a nondecrease of the total entropy is wanted, the external entropy variation cannot always be identified with the entropy $S^{\mathcal{T}m}$; this latter entropy concept rather expresses a measure of the amount of information that is contained in the probability distribution (p_i^{ω}), according to the mathematical theory of communication (Shannon and Weaver, 1949).

In a less phenomenological model than the one consisting in the family of operations $\mathcal{T} = (T_i)$, the external entropy production occurs on an external system which is supposed to interact with the open system; hence the computation of the external entropy production concerns the description of the final state of such an external system, which may be supposed to be in an initial pure state (Kraus, 1971, Section 2).

In this respect, the mathematical construction carried out within the proof of the main theorem constitutes a simple dynamical model arising from a purely quantum description of the external system: this model generalizes the well-known one occurring within the quantum theory of measurement, along the lines of the approach of Hellwig and Kraus (1969, 1970; Kraus, 1971, 1983).

The "scattering" operator $\mathcal{U}$ in (18), if it is constructed so as to be unitary, as outlined after formula (16), describes the free Hamiltonian evolution of the composite system (quantum system plus external system), from an initial state, described by $W \otimes P_1$, to a final one; afterward a perfect nondestructive measurement is performed on the composite system with respect to a complete family of orthogonal properties (P_i) of the external system. I remark that the possibility of describing the whole family $\mathcal{T} = (T_i)$ of operations as a perfect nondestructive measurement on the composite system is a consequence of the normalization condition (13).

The final reduced density operator of the external system turns out to be $\sum_i p_i^{\omega} P_i$, as follows by taking the partial trace of (18) with respect to the Hilbert space of the given quantum system. Thus, irrespective of the initial state ω of the quantum system, in this model of a pure discrete interaction of the first kind the external system is always left in a mixture of the same orthogonal pure states; hence, the description of the external system concerning its entropy evaluation may be equivalently (Herbut, 1986) accomplished through the classical state (p_i^{ω}).

Concerning pure operations of the second kind, by using their representation (7), it is easily realized that a quantum model of the type considered above can even be constructed when such operations occur; in general, it could even be constructed when nonpure completely positive operations occur (Kraus, 1971, 1983; Lindblad, 1976); however, in these cases the resulting quantum model is expected to be less simple than the previous one, so that the entropy of the reduced density operator of the external system is no longer equal to the mixing entropy $S^{\mathcal{T}m}(\omega)$: then it is not surprising that the right-hand inequality of (22) may be violated.

ACKNOWLEDGMENT

The subject of this work was suggested to me by Prof. R. Ascoli; I am very grateful to him for his encouragement and many helpful comments.

This work was done within the National Group on Mathematical Problems in Quantum Theory of the Ministero della Pubblica Istruzione at the dipartimento di Matematica del Politecnico di Torino, Torino, Italy.

REFERENCES

Ascoli, R., Urigu, R. (1984). *Nuovo Cimento*, **83B**, 201.
Brillouin, L. (1962). *Science and Information Theory.* Academic Press, New York.
Davies, E. B. (1969). *Communications in Mathematical Physics*, **15**, 277.
Davies, E. B. (1976). *Quantum Theory of Open Systems*, Academic Press, London.
Haag, R., and Kastler, D. (1964). *Journal of Mathematical Physics*, **5**, 848.
Hellwig, K. E., and Kraus, K. (1969). *Communications of Mathematical Physics*, **11**, 214.
Hellwig, K. E., and Kraus, K. (1970). *Communications of Mathematical Physics*, **16**, 142.
Herbut, F. (1986). *International Journal of Theoretical Physics*, **25**, 1215.
Klein, O. (1931). *Zeitschrift für Physik*, **72**, 767.
Kraus, K. (1971). *Annals of Physics*, **64**, 311.
Kraus, K. (1983). *Lecture Notes in Physics*, Vol. 190, Springer, Berlin.
Lindblad, G. (1976). *Communications of Mathematical Physics*, **48**, 119.
Lüders, G. (1951). *Annalen der Physik*, **8**, 322.
Ludwig, G. (1970). *Lecture Notes in Physics*, Vol. 4, Springer, Berlin.
Ludwig, G. (1972). Meß-und Präparierprozesse, in *Notes in Mathematical Physics*, Vol. 6, Marburg.
Ludwig, G. (1983). *Foundations of Quantum Mechanics*, I-II, Springer, New York.
Messiah, A. (1961-1962). *Quantum Mechanics*, I-II, North-Holland, Amsterdam.
Penrose, O. (1970). *Foundations of Statistical Mechanics*, Pergamon Press, Oxford.
Riesz, F., and Nagy, B. Sz. (1960). *Functional Analysis*, 3rd ed. Appendix, Frederic Ungar, New York.
Shannon, C., and Weaver, W. (1949).
Szilard, L. (1929). *Zeitschrift für Physik*, **53**, 840.
Von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, New Jersey.
Wehrl, A. (1978). *Review of Modern Physics*, **50**, 221.